

PRECISE SOLUTION FOR A FINITE SET OF
SPHERICAL COEFFICIENTS FROM EQUIANGULAR GRIDDED DATA

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INTRODUCTION

An important goal of geodesy is to determine the anomalous potential and its derivatives outside of the earth. Representing the surface anomalies by a series of spherical harmonics is useful since it is then possible to do a term by term solution of Laplace's equation and upward continuation. This paper addresses the problem of finding such a spherical harmonic series for anomaly values given on an equiangular surface grid. (This is a first step toward the more complicated problem of finding a function such that locally averaged values fit a grid of mean anomalies.) Three approaches to this fitting problem are discussed and compared: the discrete Fourier technique, the discrete integral technique, and a new approach by this author. The peculiar nature of the equiangular grid, with its increasing density of (noisy) data toward the poles, causes each method to exhibit a different type of difficulty. The new method is shown to be practical as well as precise since the numerical conditioning problems which appear can be successfully handled by such well-known techniques as a (simple) Kalman filter.

DISCRETE FOURIER METHOD

The discrete Fourier method [Dilts, 1985] uses a discrete Fourier series to represent both the longitude and latitude variation of the desired function. The data at the (i, j) grid point on a grid of N latitude and $2N$ longitude intervals can be uniquely represented by the double Fourier series,

$$f(\theta_i, \phi_j) = \sum_{q=-N}^N \sum_{m=-N}^N A_{qm} e^{iq\theta_i} e^{im\phi_j} . \quad (1)$$

The discrete Fourier method makes its modeling assumption at this point by choosing the function off the grid points to be given by this same double Fourier expansion. Comparison of the continuous spherical representation

$$f(\theta, \phi) = \sum_{n=0}^L \sum_{m=-N}^N C_{nm} \bar{P}_{nm}(\theta) e^{im\phi} \quad (2)$$

and expansion of the normalized Legendre polynomials

$$\bar{P}_{nm}(\theta) = \sum_{q=-n}^n B_q^{nm} e^{iq\theta} \quad (3)$$

to the function modeled as in Equation (1) then yields

$$\sum_{n=0}^L C_{nm} B_q^{nm} = \begin{cases} A_{qm} & \text{for } |q| \leq N \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

A solution exists for the upper limit L, equal to infinity. It can be expressed as

$$C_{nm} = \sum_{q=-N}^N Z_q^{nm} A_{qm} \quad (5)$$

where the "inverse" coefficients are obtained from

$$\frac{\pi}{2} \bar{P}_{nm}(\theta) |\sin\theta| = \sum_{q=-\infty}^{\infty} Z_q^{nm} e^{-iq\theta} \quad (6)$$

for θ between zero and 2π radians.

The shortcoming of this approach is the need for an infinite number of terms to solve Equation (4) for arbitrary A_{qm} (representing the data). Small amounts of noise in A_{qm} can lead to the presence of terms in the double Fourier expansion (Eq. (1)) which are not present in the gravity field and which have infinite derivatives at the poles. Truncation of the series is the strategy for coping with this difficulty. After truncation, the function will no longer match the gridded data, and the degree of discrepancy is not under the analyst's control.

DISCRETE INTEGRAL METHOD

The discrete integral approach has been widely used (see for example Colombo [1981]). It approximates the continuous inversion integral for the spherical coefficients by a discrete, weighted sum.

$$C_{nm} = \sum_{i=0}^N \sum_{j=0}^{2N-1} \bar{P}_{nm}(\theta_i) e^{-im\phi_j} W_i f(\theta_i, \phi_j) \quad (7)$$

The weights W_i are usually chosen to be the grid block areas. The difficulty with this approach is that the discrete $\bar{P}_{nm}(\theta_i)$ are not orthogonal on the equiangular grid. As a result, aliasing occurs, and the resultant spherical expansion does not match the gridded data. The expansion is truncated at degree N or less, and the amount of the discrepancy is thus only indirectly under the analyst's control. Comparison with the preceding technique is obtained by using the expansion of Equation (6) in the above expression (with the weights proportional to area and the interval extended to 2π):

$$C_{nm} = \sum_{q=-N}^N (Z_q^{nm} + Z_{q+2N}^{nm} + Z_{q-2N}^{nm} + \dots) A_{qm} + \text{pole terms.} \quad (8)$$

Comparison with Equation (5) shows that it corresponds to the leading term in the above expression. Thus, taking the degree of the discrete integral expansion to infinity does not appear to reproduce the gridded data.

NEW METHOD

The third method is newly presented by this author. It uses the Fourier representation of the data (Equation (1)) but makes its modeling assumption in the spherical domain. Comparison to the spherical expansion only at the grid points yields

$$\sum_{n=0}^L C_{nm} (B_q^{nm} + B_{q+2N}^{nm} + B_{q-2N}^{nm} + \dots) = A_{qm}. \quad (9)$$

This differs from Equation (4) since it is the result of a discrete comparison at the grid points (using the periodic nature of the discrete exponential) and not a comparison of continuous functions. If L is chosen equal to $N+|m|-2$ (except $L=N$ for $m=0$), Equation (9) then becomes an invertible matrix equation (with E indicating the sum of the B terms):

$$EC = A \quad \text{and then} \quad C = E^{-1}A. \quad (10)$$

Since the inverse yields a precise fit at the data points, the modeling assumption is that all the C_{nm} 's are zero for n greater than $N+|m|-2$. The continuous function resulting from using these C 's in a spherical expansion thus reproduces the data and has a finite number of terms. Since $L < 2N$ the elements of the matrix E are easy to compute: at most two of the B terms in Equation (9) are non-zero. Even for terms of degree less than N , this solution is different from the discrete Fourier case, Equation (5), since (ZE) is not the identity.

The difficulty with this method is that the matrix E becomes ill-conditioned for large values of the order m . There are, however, many well-known and trustworthy techniques for dealing with such problems. A few such techniques are summarized below.

- o Perform the transformation of E to the identity in a column by column fashion, stopping when the conditioning becomes a problem. If this process is stopped at the column for degree N , the discrete Fourier approximation is obtained. Further steps toward finding E^{-1} constitute improved approximations.

- o Invert the matrix $(E+\delta I)$ for a small δ and use it instead of E^{-1} .
- o Use a simple Kalman filter

$$A = EC + V; \quad C = E^T(EE^T + \gamma I)^{-1}A \quad (11)$$

where the measurement noise, V , has variance $\gamma_1 I$ and the prior uncertainty on C is $\gamma_2 I$. Then $\gamma = \gamma_1/\gamma_2$ and is a small quantity.

- o Use a more complicated Kalman filter with detailed models for the noise and for the initial uncertainty.

All of these strategies yield results which are not overly sensitive to noise. By adjusting the parameters in these methods, the analyst can control how close the reconstructed function comes to the gridded data (allowing only for small deviations consistent with the noise model). Use of the Kalman filters also has the advantage of providing uncertainties in the estimated spherical coefficients.

SUMMARY

The problem of fitting a smooth function to data given on an equiangular spherical grid has been discussed. Two existing approaches were summarized and a new approach was presented. Each approach was found to possess an area of difficulty resulting from the properties of the equiangular grid. Well-known techniques (such as Kalman filtering) are available as practical strategies for dealing with the numerical conditioning in the new method. As a result, the new method is practical and capable of reproducing the gridded data to a precision consistent with the noise model.

REFERENCES

- Colombo, O. L., Numerical Methods for Harmonic Analysis on the Sphere, Ohio State University Department of Geodetic Science Report 310, 1981.
- Dilts, G. A., Computation of Spherical Harmonic Coefficients via FFT's, J. of Computational Physics, 57, 439-453, 1985.